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DYNAMICS AT NON-DIFFERENTIABLE SCALE IN THE MULTIFRACTAL THEORY OF MOTION

BY

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Abstract. Using the Multifractal Theory of Motion, the solutions of a Navier-Stokes stationary system at non-differentiable scale resolutions, are given. The solutions corresponding to such a system are non-linear, in the form of multifractal solitons and multifractal soliton-kink mixtures

Keywords: soliton; kink; Multifractal Theory of Motion.

1. Introduction

Both in the context of Scale Relativity Theory (Nottale, 2011), as well as in the one of Multifractal Theory of Motion (Mercheș and Agop, 2016),

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assuming that any physical system is equated both structurally and operationally to a multifractal object, said system's dynamics can be outlined using motions of its structural units, dependent on the chosen scale resolution, on continuous but non – differentiable curves (multifractal curves). Since for a substantial temporal scale resolution in relation to the inverse of the largest Lyapunov exponent (Politi and Pikovsky, 2016), the deterministic bearings of any structural units belonging to the physical system, can be superseded by a bundle of potential (“virtual”) trajectories, the idea of certain trajectory can be substituted with the notion of probability density.

With all of the above considerations taken into account, the multifractality expressed through stochasticity, in the expression of the dynamics of any physical system, becomes operational. This implies that, in the description of the dynamics of any physical system, instead of “operating” with a singular variable expressed through a precise non – differentiable function, it is feasible to “operate” only with estimations of said mathematical function, acquired by extracting their average values on various scale resolutions. Consequently, any variable purposed to describe the physical system dynamics will do as such, in the guise of being the limit of a cluster of mathematical functions, this serving as non – differentiable for null scale resolutions and differentiable in other cases (Nottale, 2011).

2. Short Reminder on the Multifractal Theory of Motion

The fundamental hypothesis of the Multifractal Theory of Motion postulates that the dynamics of any physical system are outlined by means of the multifractal curves. This induces several consequences (Nottale, 2011; Mercheş and Agop, 2016):

(i) Any multifractal curve is expressly scale δt dependent. In detail, its length aims towards infinity when δt aims towards zero (Lebesgue theorem) (Mandelbrot, 1982). Additionally, the space develops into a multifractal, in the Mandelbrot sense;

(ii) The dynamics of the physical system are related to the functionality of a bundle of functions during the zoom operation of δt . Then, $\delta t \equiv dt$ by means of the operation of the substitution principle;

(iii) The dynamics of the physical structural units are expressed by means of multifractal variables. Then, two derivatives of the variable range $Q(t, dt)$ can be defined:

$$\begin{aligned} \frac{dQ_+}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{Q(t, t + \Delta t) - Q(t, \Delta t)}{\Delta t}, \\ \frac{dQ_-}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{Q(t, \Delta t) - Q(t - \Delta t, \Delta t)}{\Delta t}. \end{aligned} \tag{1}$$

The sign “+” is linked to the forward processes, while the sign “-” is linked to the backward processes.

(iv) The differential belonging to the spatial coordinate field takes the form:

$$d_{\pm}X^i(t, dt) = d_{\pm}x^i(t) + d_{\pm}\xi(t, dt) \quad (2)$$

The differentiable part $d_{\pm}x^i(t)$ is independent of the scale resolution, but the non – differentiable part $d_{\pm}\xi(t, dt)$ is dependent of the scale resolution.

(v) The non – differentiable part of the spatial coordinate range fulfills the non – differentiable equation

$$d_{\pm}\xi^i(t, dt) = \lambda_{\pm}^i(dt)^{\left[\frac{2}{f(\alpha)}\right]-1} \quad (3)$$

where λ_{\pm}^i are constant coefficients linked to differentiable – non – differentiable shift, $f(\alpha)$ is the singularity spectrum of magnitude α of fractal dimension and α is the singularity index. A multitude of modes exists, and as such, a diverse assortment of definitions concerning fractal dimensions: more to the point, the fractal dimension as defined by Kolmogorov, the fractal dimension as defined by Hausdorff – Besikovici etc. (Mandelbrot, 1982). Picking out one such function and “working” in the physical system, the value of the fractal dimension has to be constant and arbitrary for the totality of the dynamical analysis. For instance, it is frequently encountered that $D_F < 2$ for correlative physical processes, $D_F > 2$ for non – correlative physical processes etc. In such a context, by employing (3), there is a possibility to discern not just the “areas” of the physical system dynamics that are described through a particular fractal dimension, but also the range of “areas” for which their fractal dimensions are positioned in an interval of values. Moreover, by means of the singularity spectrum $f(\alpha)$, it is possible to distinguish classes of universality in the physical system dynamics laws, even in the case where regular or strange attractors display various aspects (Agop and Mercheș, 2019).

(vi) The differential time reflection invariance of any one variable is retrieved with the help of the operator:

$$\frac{\hat{d}}{dt} = \frac{1}{2} \left(\frac{d_+ + d_-}{dt} \right) - \frac{i}{2} \left(\frac{d_+ - d_-}{dt} \right). \quad (4)$$

This is an innate result of Cresson’s theorem (Nottale, 2011). Using the operator (4) to X^i , the complex velocity field is obtained:

$$\hat{V}^i = \frac{\hat{d}X^i}{dt} = V_D^i - V_F^i \quad (5)$$

with

$$V_D^i = \frac{1}{2} \frac{d_+ X^i + d_- X^i}{dt}, \quad V_F^i = \frac{1}{2} \frac{d_+ X^i - d_- X^i}{dt}, \quad i = 1, 2, 3 \quad (6)$$

The real part of V_D^i is independent with respect to the scale resolution, while the imaginary one V_F^i is dependent with respect to the scale resolution.

(vii) Because multifractalization involves stochasticization (Nottale, 2011), the whole statistic “collection”, found as averages, variances, covariances etc., manifests operationally. As such, let the functionality linked to the average of $d_{\pm} X^i$ be described as:

$$\langle d_{\pm} X^i \rangle \equiv d_{\pm} x^i, \quad (7)$$

with

$$\langle d_{\pm} \xi^i \rangle = 0, \quad (8)$$

The above relation (8) asserts that the average related to the non – differential part belonging to the spatial coordinate field is null.

(viii) The physical system dynamics can be expressed by means of the scale covariant derivative stated by the operator (9) and (10):

$$\frac{\hat{d}}{dt} = \partial_t + \hat{V}^i \partial_i + \frac{1}{4} (dt)^{\left[\frac{2}{f(\alpha)}\right]-1} D^{lk} \partial_l \partial_k, \quad (9)$$

where

$$D^{lk} = d^{lk} - i \bar{d}^{lk}, \quad d^{lk} = \lambda_+^l \lambda_+^k - \lambda_-^l \lambda_-^k, \quad \bar{d}^{lk} = \lambda_+^l \lambda_+^k + \lambda_-^l \lambda_-^k \quad (10)$$

For physical Markov – type stochastic processes,

$$\lambda_+^i \lambda_+^l = \lambda_-^i \lambda_-^l = 2\lambda \delta^{ie}, \quad (11)$$

and for

$$f(\alpha) \equiv D_F \quad (12)$$

where λ is a specific coefficient linked to the fractal – non – fractal shift, the scale covariant derivation becomes:

$$\frac{\hat{d}}{dt} = \partial_t + \hat{V}^l \partial_l - i\lambda (dt)^{\left[\frac{2}{D_F}\right]-1} \partial_l \partial^l \quad (13)$$

When looking at the distinct case of motions on Peano – type curves, which implies $D_F = 2$, the scale covariant derivative (13) can be expressed in the regular form, from the Scale Relativity Theory (Nottale, 2011):

$$\frac{\hat{d}}{dt} = \partial_t + \hat{V}^l \partial_l - iD \partial_l \partial^l \quad (14)$$

where $\lambda \equiv D$ is the diffusion coefficient linked to fractal – non – fractal shift. As such, this model capitalizes (in a general sense) all the results of Nottale's theory (*i.e.* Scale Relativity Theory) (Nottale, 2011).

Now, taking into account the functionality related to the scale covariance principle, *i.e.* employing the operator (9) for the complex velocity fields (5), in the lack of any external constraint, the following form related to the motion equations (*i.e.* the geodesics equation explained on a multifractal space) is expressed:

$$\frac{\hat{d}\hat{V}^i}{dt} = \partial_t \hat{V}^i + \hat{V}^l \partial_l \hat{V}^i + \frac{1}{4} (dt)^{\left[\frac{2}{f(\alpha)}\right]-1} D^{lk} \partial_l \partial_k \hat{V}^i = 0, \quad (15)$$

This implies that the multifractal acceleration, $\partial_t \hat{V}^i$, the multifractal convection, $\hat{V}^l \partial_l \hat{V}^i$ and the multifractal dissipation $D^{lk} \partial_l \partial_k \hat{V}^i$ achieve their equilibrium in any point belonging to the multifractal curve. In particular, for (11) and (12), the motion Eq. (15) is expressed as:

$$\frac{\hat{d}\hat{V}^i}{dt} = \partial_t \hat{V}^i + \hat{V}^l \partial_l \hat{V}^i - i\lambda (dt)^{\left[\frac{2}{D_F}\right]-1} \partial_l \partial^l \hat{V}^i = 0 \quad (16)$$

3. Navier-Stokes-Type Equation at Non-Differentiable Scale

By separating in the motion Eq. (16) the dynamics of any physical system, on scale resolutions, the following equations are obtained:

$$\partial_t V_D^i + V_D^l \partial_l V_D^i - \left[V_F^l + \lambda (dt)^{\left[\frac{2}{f(\alpha)}\right]-1} \partial^l \right] \partial_l V_F^i = 0 \quad (17)$$

at differentiable scale resolution, and:

$$\partial_t V_F^i + V_D^l \partial_l V_F^i + \left[V_F^l + \lambda (dt)^{\left[\frac{2}{f(\alpha)}\right]-1} \partial^l \right] \partial_l V_D^i = 0, \quad (18)$$

at non-differentiable scale resolution.

Because in the dynamic analysis only the non-differentiable scale behaviors are of interest, then in the [(17), (18)] system of equations, the following condition must be imposed:

$$V_D^i = 0 \quad (19)$$

Then, with these restraints, for any physical system with a constant state density $\varphi = \text{const.}$, in the static case, the Navier-Stokes-type system is obtained:

$$\left[V_F^l + \lambda(dt)^{\left[\frac{2}{f(\alpha)}\right]-1} \partial^l \right] \partial_l V_F^i \equiv 0 \quad (20)$$

$$\partial_i V_F^i = 0 \quad (21)$$

These differential equations written by means of dimensionless plane coordinates, with suitable initial and boundary conditions admit the following solutions (Agop and Mercheş, 2019):

$$U = \frac{1.5}{(\nu\xi)^{\frac{1}{3}}} \operatorname{sech}^2 \left[\frac{0.5\eta}{(\nu\xi)^{\frac{2}{3}}} \right] \quad (22)$$

$$V = \frac{1.9}{(\nu\xi)^{\frac{1}{3}}} \left\{ \frac{\eta}{(\nu\xi)^{\frac{2}{3}}} \operatorname{sech}^2 \left[\frac{0.5\eta}{(\nu\xi)^{\frac{2}{3}}} \right] - \tanh \left[\frac{0.5\eta}{(\nu\xi)^{\frac{2}{3}}} \right] \right\} \quad (23)$$

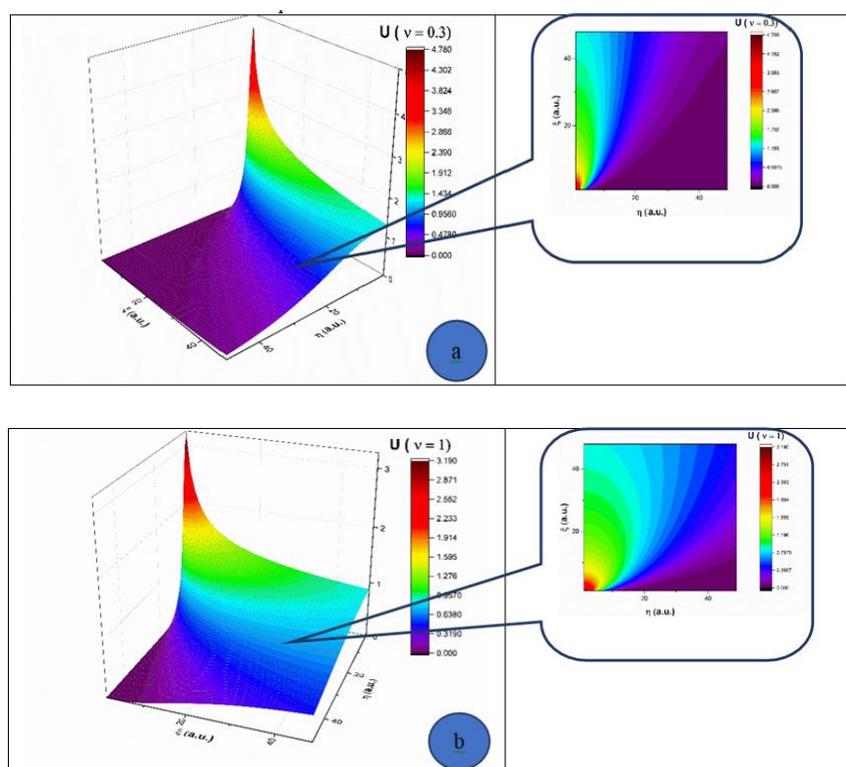
where ξ and η are nondimensional spatial coordinates, U and V are the nondimensional components belonging to the velocity field along the $O\xi$ and $O\eta$ axes, and ν is the multifractality degree.

As such, the velocity field along the $O\xi$ axis is described by the multifractal soliton (14), while the velocity field along the $O\eta$ axis is described by the multifractal soliton – kink (15).

In such a context, when investigating the dynamic of a complex fluid expansion in a multifractal medium, there are two types of scales that need to be taken into account. Firstly, there are the internal interaction scales, which is an amalgam of dynamics induced by the properties of the complex fluid and by its nature. For example, if the complex fluid is considered as a multi element transient plasma (Irimiciuc *et al.*, 2018), this scale will be dominated by collision, chemical processes, molecular formation, ionization processes, excitations etc. The external interaction scales contain the dynamic between the complex fluid and the multifractal medium in which the fluid is embedded. Keeping the same example as before for the plasma as a complex fluid, this scale can relate to the overall dynamics of the plasma, gas-plasma interactions or plasma confinement. These interactions can also be investigated on an interface separating the two fractal objects meaning one could potentially investigate just the double layer separating flowing transient plasma and the background gas and explore all the phenomena mentioned before.

In the following, let the influence of the multifractality degree on each of the two components (U and V) of the complex fluid for a 2D flow be

explored. In Fig. 1 in 3D and contour plot are represented the velocity component (U) on the $O\xi$ for three multifractality degrees (0.3, 1 and 3). For a low multifractality degree it is noticed a very directional flow mainly across the $O\xi$ with little spatial expansion. The enhancement of the multifractality in the system leads to a decrease of the velocity and a strong lateral expansion. It is important to note that the main expansion direction does not change, only the contributions on the $O\eta$ direction. The multifractality degree of the system on this velocity component acts as multifractal-like dispersion phenomena. In Fig. 2 in 3D and contour plot are represented the velocity component (V) on the $O\eta$ for three multifractality degrees (0.3, 1 and 3). Let it be noted that this component of the velocity is not influenced by the multifractality degree when investigating the absolute value of the velocity, thus remaining quasi constant. There is however a strong influence on the direction of the component. For a low multifractality degree there is a small angle with respect to the $O\xi$ axis. Higher values of multifractality induce a change in the expansion angle transitioning towards higher angles. The multifractality degree of the complex fluid on this velocity component works towards the uniformization of the V component as the distribution tends to reach the maximum expansion velocity available for the complex fluid.



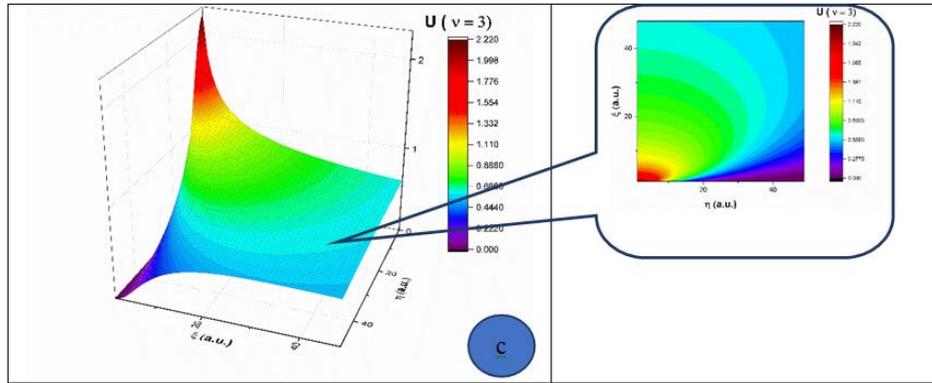
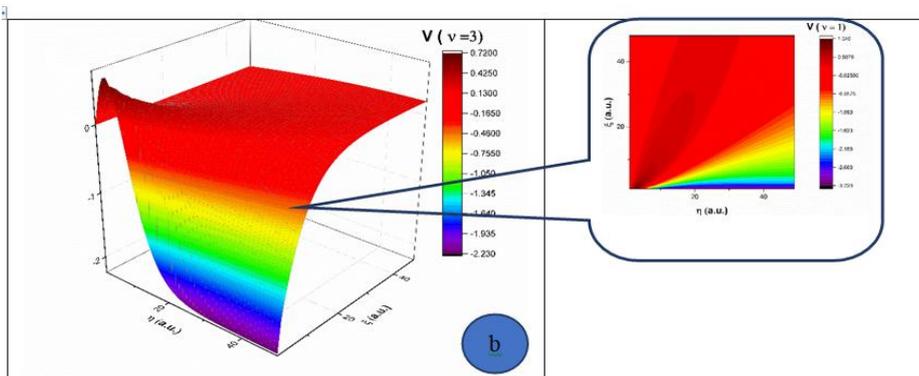
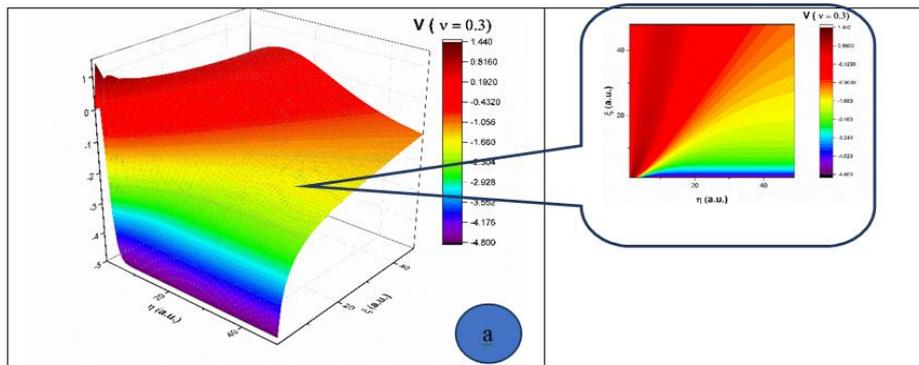


Fig. 1 – 3D and contour plot representation of the velocity component on the $O\xi$ for three multifractality degrees: *a*) 0.3, *b*) 1, and *c*) 3.



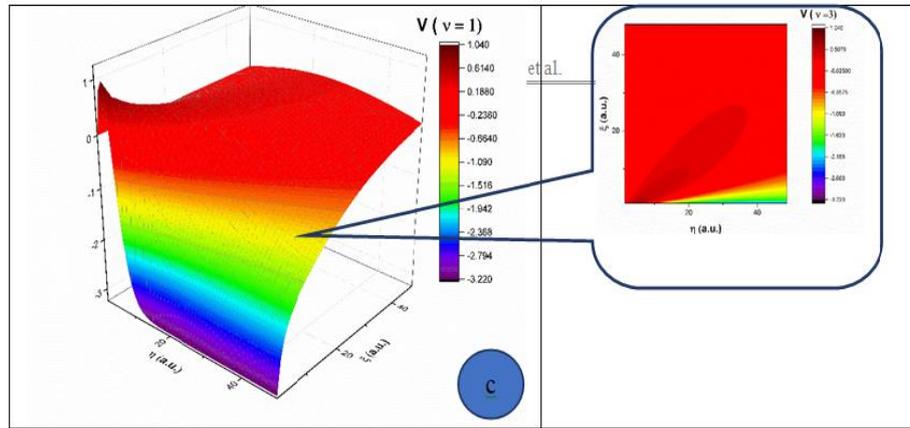
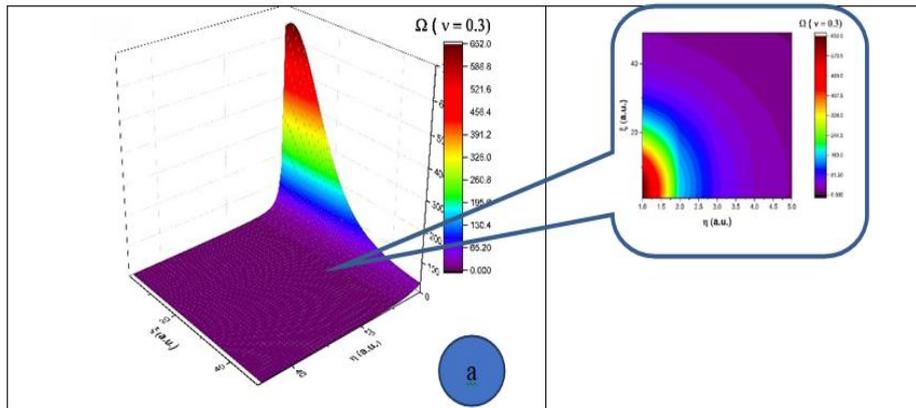


Fig. 2 – 3D and contour plot representation of the velocity component on the Ω_η for three multifractality degrees: *a)* 0.3, *b)* 1, and *c)* 3.

Let it be noted that a multifractal minimal vortex can be associated to the velocity field given through (22) and (23):

$$\Omega = (\partial_\eta U - \partial_\xi V) = \frac{0.57\eta}{(v\xi)^2} + \frac{0.63\xi}{(v\xi)^{\frac{4}{3}}} \tanh\left[\frac{0.5\eta}{(v\xi)^{\frac{2}{3}}}\right] + \frac{1.9\eta}{(v\xi)^2} \operatorname{sech}^2\left[\frac{0.5\eta}{(v\xi)^{\frac{2}{3}}}\right] - \frac{0.57\eta}{(v\xi)^2} \tanh^2\left[\frac{0.5\eta}{(v\xi)^{\frac{2}{3}}}\right] - \left[\frac{1.5}{v\xi} + \frac{1.4\eta}{\xi(v\xi)^{\frac{5}{3}}}\right] \operatorname{sech}^2\left[\frac{0.5\eta}{(v\xi)^{\frac{2}{3}}}\right] \tanh\left[\frac{0.5\eta}{(v\xi)^{\frac{2}{3}}}\right] \quad (24)$$

In Fig. 3 are presented the 3d and contour plots of the multifractal minimal vortex.



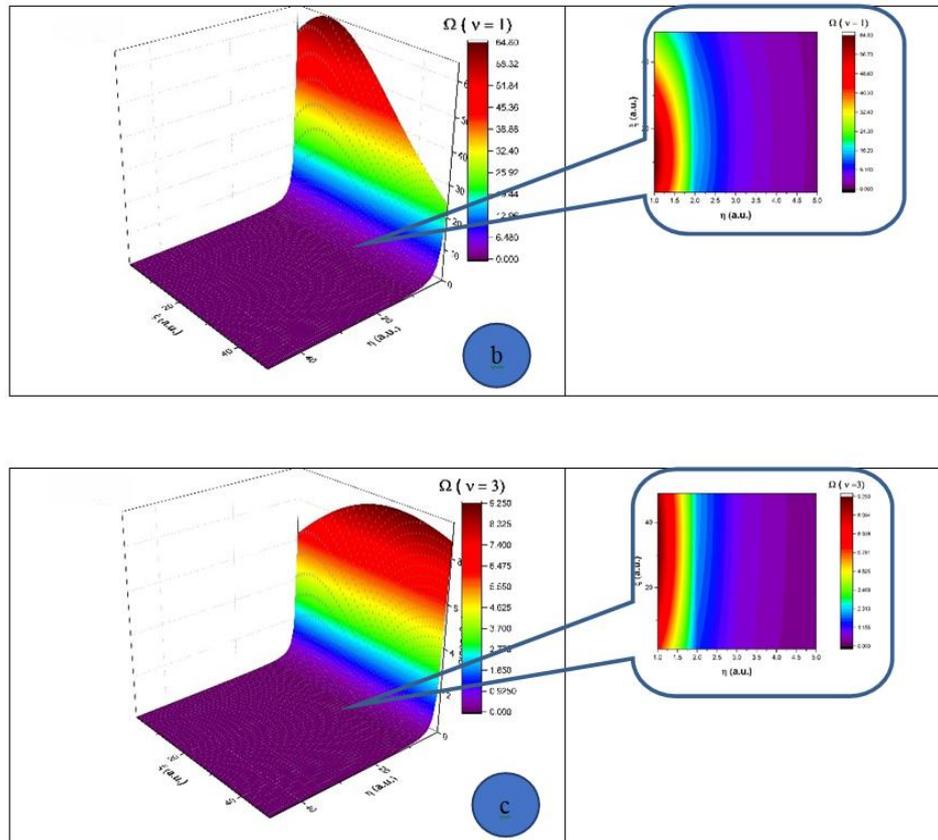


Fig. 3 – 3D and contour plot representation of the multifractal minimal vortex or three multifractality degrees: a) 0.3, b) 1, and c) 3.

4. Conclusions

As such, the multifractal soliton (22) and the soliton-kink multifractal mixture (23) are liable, by means of the multifractal minimal vortex (24), for the management of turbulences at non-differentiable resolutions of scale. Despite the fact that they are non-manifested when referring to differentiable scale resolution, these turbulences have the potential to become manifest at the same scale by means of the “synchronization” (self-structuring) of multifractal minimal vortices in the form of vortices streets.

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DINAMICI LA SCARĂ NEDIFERENȚIABILĂ ÎN TEORIA MULTIFRACTALĂ A MIȘCĂRII

(Rezumat)

Utilizând Teoria Multifractală a Mișcării, se obțin soluții ale unui sistem de ecuații diferențiale de tip Navier-Stokes staționar, la rezoluție de scară nediferențabilă. Soluțiile unui asemenea sistem sunt puternic neliniare, de forma solitonului multifractal și mixturii soliton – kink multifractal.

